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VIBRATIONAL POWER FLOW ANALYSIS OF DAMAGED BEAM STRUCTURES

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In view of structure-borne sound, vibrational power flow analysis is investigated for damaged beam structures. In this paper, the damage is modelled as a joint of a local spring. The damage point transfer matrix and the beam element transfer matrix are deduced, then the relations of the vibrational power flow, the position and the characteristic size of the damage are obtained combined with periodic structure theory. Based on the theoretical analysis and measurement results, the damage can be diagnosed in the next work.

1. INTRODUCTION

Any structure composed of a number of identical elements which are connected in a regular pattern is said to be "periodic". For example, periodic beam structure is one of the most basic structure types to support the shells or fuselage of ships or aircraft. Under fatigue load and external pulse conditions, the damages may be produced as a result of the flaws or manufacturing defects in the structures. So dangers are inherent in the life of the structures. For this reason, methods for making early detection and location of damages have been the subject of recent investigation.

Most of the previous researchers on vibration-related damage detection are based on modal methods. The basis for such methods is that damage produces a decrease in dynamic stiffness, this decrease in turn produces decreases in natural frequencies for an undamped simple beam. This basic premise has produced a number of results using modal analysis, i.e., frequency measurement to perform diagnostics [1]. Though modal-based method may have advantages, modal-based method possesses a number of major disadvantages [2]. First of all, some of the modal-based method investigations provide a strong argument for including the geometry of the damage in any diagnostic testing scheme, something which is not easily done in frequency-based methods. Indeed, mode and frequency characterizations are not so simple in variable structure systems; there is ample evidence that one should not use modal methods based on uniform undamped simple beams or plates as is often done in the engineering literature in addressing damage assessment methodologies. Since material parameters are most properly considered as spatially dependent quantities with damage manifested by changes in geometry (and hence in the spatial dependence of these parameters), it is unlikely that any rigorous theoretical basis for modal-based method for variable material structures would emerge. But perhaps the most serious objection to modal-based method resides in the fact that modal-based methods have been shown to be highly unreliable for estimation of variable material parameters such as damping in composite material structures.

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The results of previous efforts in damage detection provide evidence that something is gained by including the effects of geometry and hence modelling the local changes in modulus. This then raises significant questions as to the validity of using traditional modal analysis (i.e., measurements of natural frequencies, assuming a uniform model) as the foundation of a damage detection.

One of the major concerns regarding using modal analysis to diagnose damage is that damages is a local phenomenon and modal information is a reflection of the global system properties. If the damage is located on a nodal point of a certain mode, the corresponding frequencies would be all the same for all sizes of damage and the same in turn as those for no damage. The nodal point of a mode shape can be taken as a “point of inflection” at which the influences of damage vanish. Weissenburger’s study [3] combined with Sato’s result [4] indicates a possible source of why the previous work contains conflicting statements on the use of frequency measurements to predict damage. An explanation for the controversy may be simple, since it depends on where the damage is located, i.e., it depends on the geometry of the damage. For some types of damage modal analysis may be appropriate while for other configurations it may not be.

In recent years, the structure-borne sound analysis and control of flexible structures and cabins of marine structures and aeronautical crafts are becoming an important topic. The use of vibrational power flow in a problem of this type is very valuable. An attempt to decrease the radiation or vibration in a structure by reducing only the force or velocity amplitude and not considering the relative phase angle may not necessarily be successful, but an improvement may be ensured by decreasing the net vibrational power applied to a structure [5]. The premise of the effort proposed here is that damage of a structure will correspond in some way to changes, though small, in the structure’s mass, damping and stiffness properties and so the vibrational power flow is influenced by the changes in propagating waves. Through the study of vibrational power flow, the major structure-borne sound source is not only identified [6], but also the position and size of damage can be diagnosed.

In the view of structure-borne sound, vibrational power flow analysis is investigated for damaged beam structures. Damages in a structure cause changes in the physical coefficients of mass density, elastic modulus and damping coefficients. In this paper, the damage is modelled as a joint of a local spring. The damage point transfer matrix and the beam element transfer matrix are deduced, then the relations of the vibrational power flow, the position and the characteristic size of the damage are obtained combined with periodic structure theory. Further work will be done to diagnose the damage based on the above analysis.

2. FREE VIBRATION OF EULER–BERNOULI BEAM

The Euler–Bernouli equation of free vibration is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho S\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where E is Young’s modulus, I is the moment of inertia, ρ is the material density, S is the area of the cross-section and w is the flexural displacement.

For harmonic motion $w = W(x)e^{i\omega t}$, equation (1) can be written as (for brevity, the term $e^{i\omega t}$ is ignored in the all following formulas)

$$\frac{d^4 W}{dx^4} - \lambda^4 W = 0, \tag{2}$$

where $\lambda^4 = \omega^2 \rho S/EI$ and ω is the angular frequency.

When introducing state vector $V(x)$, the solution of equation (2) is [7]

$$V(x) = U(x)V(0), \tag{3}$$

where

$$V(x) = \begin{Bmatrix} W(x) \\ \theta(x) \\ M(x) \\ Q(x) \end{Bmatrix}, \qquad U(x) = \begin{bmatrix} u_1 & u_2/\lambda & u_3/EI\lambda^2 & u_4/EI\lambda^3 \\ u_4\lambda & u_1 & u_2/EI\lambda & u_3/EI\lambda^2 \\ u_3EI\lambda^2 & u_4EI\lambda & u_1 & u_2/\lambda \\ u_2EI\lambda^3 & u_3EI\lambda^2 & u_4\lambda & u_1 \end{bmatrix}, \tag{4}$$

with

$$u_1 = [\cosh(\lambda x) + \cos(\lambda x)]/2, \qquad u_2 = [\sinh(\lambda x) + \sin(\lambda x)]/2,$$

$$u_3 = [\cosh(\lambda x) - \cos(\lambda x)]/2, \qquad u_4 = [\sinh(\lambda x) - \sin(\lambda x)]/2 \tag{5}$$

and W_0, θ_0, M_0, Q_0 in equation (3) are the transverse deflection, the slope, the bending moment and the shear force of the cross-section $x = 0$ respectively. $U(x)$ is the transfer matrix of beam element.

3. TRANSFER MATRIX OF PROPAGATION WAVE

A model of periodically simply supported beam is shown in Figure 1. Every element is influenced by its adjacent elements and it can be simplified as a simply supported beam with two bending moments applied at its ends. The state vector expression of the model is given in equation (3).

If there is a damage on a beam (shown in Figure 2), its local flexibility for typically edged damaged will produce additional coupled terms such as bending with longitudinal, torsional with transverse shear. However, it is shown [8] that if the bending moment is dominant in the beam, all other coupling terms may be neglected, and hence only the bending flexibility is considered here.

The beam has a diameter $D = 2R$, and a transverse damage of depth a. The dimensionless local flexibility of the damage for bending moment in the ξ direction is obtained as [9]

$$\bar{C} = [\pi R^3 E/(1 - \nu^2)]\, C, \tag{6}$$

where

$$C = \frac{1 - \nu^2}{E} \int_{-b}^{b} \int_{0}^{\eta} \frac{32}{\pi^2 R^8} (R^2 - \xi^2)\pi\eta F_2^2(\eta/h)\, d\eta\, d\xi, \tag{7}$$

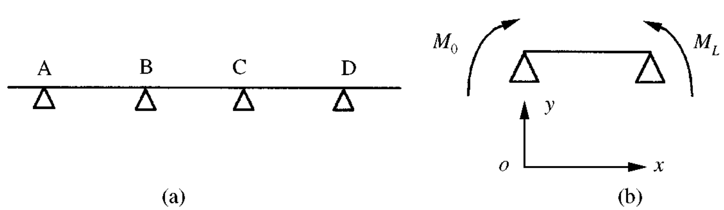

Figure 1. (a) Infinitely damaged beam on simply equi-span supports. (b) Mechanical model of element.

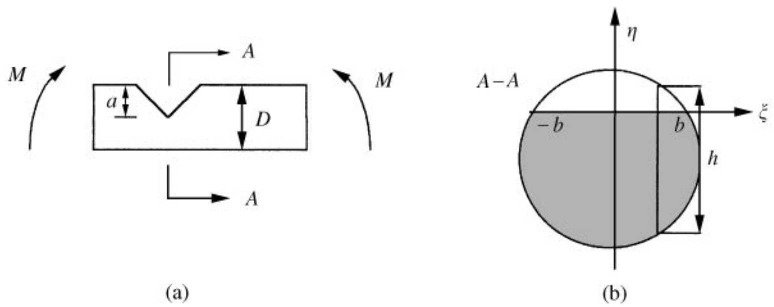

Figure 2. (a) A damaged beam in bending moment. (b) The damaged section of the beam.

v is the Poisson ratio and the geometric function $F_2(\eta/h)$ is [10].

$$F_2(\eta/h) = 1{\cdot}125 + 1{\cdot}4(\eta/h) + 7{\cdot}33(\eta/h)^2 - 13{\cdot}08(\eta/h)^3 + 14{\cdot}0(\eta/h)^4. \tag{8}$$

Assume there is a damage like the above type at $x = x_0$ in element co-ordinate system xoy. In the previous discussion, the damage joint has been modelled as a local flexibility which can essentially be regarded as a bending spring. Because of the requirement of continuity at the damage joint, the following conditions should be satisfied [1]:

$$W_r = W_l, \qquad \theta_r = \theta_l + CEI\theta_l', \qquad M_r = M_l, \qquad Q_r = Q_l, \tag{9}$$

where the subscripts r, l express the right and left sections of the damage respectively. The term $CEI\theta_l'$ is the additional slope angle resulting from the bending moment applied to the equivalent spring of the damage and a prime indicates differentiation with respect to x. In the matrix form these equations become

$$\begin{Bmatrix} W \\ \theta \\ M \\ Q \end{Bmatrix}_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & C & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} W \\ \theta \\ M \\ Q \end{Bmatrix}_l \tag{10}$$

or

$$V_r(x_0) = R_0 V_l(x_0), \tag{11}$$

where R_0 is the transfer matrix of a damage joint. For the beam element with damage shown in Figure 2, the state vector relating the values at the right end of the element ($x = L$) to those values at the left end are found:

$$V(L) = U(L - x_0) R_0 U(x_0) V(0). \tag{12}$$

4. PERIODIC SOLUTION OF BOUNDARY BENDING MOMENT

The theory of periodic structures has been established by several researchers. Loosely speaking, it is based on the assumption that harmonic motion at any point in the periodic element is equal to $\exp(\mu)$ times the motion at corresponding point in the next element, when the structure vibrates in one of the possible free waves. Here the complex number

$\mu(\mu = \mu_r + i\mu_i)$ is the propagation constant, which describes the phase change and the decay rate over the length of the periodic unit. This assumption is perfectly valid for infinite systems. According to the properties of periodically simply supported beams, there is only a value of the propagation constant if adjacent elements are coupled with a parameter. The formula for the propagation constant is [11]

$$\cos\mu = -\frac{\alpha_{0'0'}}{\alpha_{0'L'}}, \tag{13}$$

where α_{ab} (seen in reference [12]) is the receptance which is the harmonic response at point b due to unit harmonic force at point a.

Assume there is a damage in element BC and is not located at point B or point C. The bending moment M_L of point C can be solved from equation (12) if the bending moment of point B is M_0. Considering free bending wave motion from A to D, only incident bending wave exists in the right section of the damage, according to the periodic condition, the bending moments at support D is $M_L e^{\mu}$. Because of the presence of the damage, the bending moment at support B contains incident and reflected waves:

$$M_0 = M_{0P} + M_{0R}, \tag{14}$$

where M_{0P} and M_{0R} are the amplitude of incident wave and reflected wave at support B respectively. When there is no crack, only incident wave M_{0P} exists and it can be obtained by solving perfect periodic beam. That is to say, if the bending moment at support B is M_P for perfect periodic beam, $M_{0P} = M_P$. It implies that M_{0P} is known and only M_{0R} is caused by a crack here.

Based on the periodic condition, the bending moment at support A is

$$M_A = M_{0P}e^{-\mu} + M_{0R}e^{\mu}, \tag{15}$$

The slope at support B of the element AB may be found by the receptance method:

$$\theta_B = \alpha_{L'L'}M_0 + \alpha_{L'0'}M_A. \tag{16}$$

The slopes at the left support B and the right support C of the element BC are, respectively,

$$\theta_{B'} = \theta_0, \qquad \theta_C = \theta_L. \tag{17}$$

The slope at support C of the element CD is

$$\theta_{C'} = -\alpha_{0'0'}M_L - \alpha_{0'L'}M_L e^{\mu}. \tag{18}$$

Considering the continuity of slope across the supports B and C yields

$$\theta_B = \theta_{B'} = \theta_0, \qquad \theta_C = \theta_{C'} = \theta_L, \tag{19}$$

According to linear equation (12), one can assume that ($W_0 = W_L = 0$ for simply supported ends)

$$0 = a_1\theta_0 + b_1M_0 + c_1Q_0, \qquad \theta_L = a_2\theta_0 + b_2M_0 + c_2Q_0, \qquad M_L = a_3\theta_0 + b_3M_0 + c_3Q_0, \tag{20}$$

where a_i, b_i, c_i $(i = 1, 2, 3)$ are constant coefficients when x_0 and a/D are fixed. The analytical expressions of $\theta_0 = f_\theta(M_P)$, $M_{0R} = f_{0R}(M_P)$, $Q_0 = f_Q(M_P)$, etc. which are very complex may be found from equations (14)–(20). In this paper, they are solved with numerical computation, so a matrix equation can be deduced as follows:

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & -1 \\ 0 & 0 & 0 & 1 & \alpha_{0'0'} + \alpha_{0'L'}\mathrm{e}^{\mu} \\ 1 & -\alpha_{L'L'} - \alpha_{L'0'}\mathrm{e}^{\mu} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_0 \\ M_{0R} \\ Q_0 \\ \theta_L \\ M_L \end{Bmatrix} = \begin{Bmatrix} \theta_0 \\ M_{0R} \\ Q_0 \\ 0 \\ (-\alpha_{L'L'} - \alpha_{L'0'}\mathrm{e}^{-\mu})M_P \end{Bmatrix}. \tag{21}$$

5. VIBRATIONAL POWER FLOW ANALYSIS

Because only bending moment is coupled at all supports, the transmitted power flow by the bending moment M_L from the right support of the damaged beam element to its adjacent element CD is

$$P_C = \tfrac{1}{2}\mathrm{Re}(-\,\mathrm{i}\omega M_L \theta_L^*), \tag{22}$$

where Re(∗) expresses the real part of a complex value and an asterisk denotes the complex conjugate.

The transmitted power flow by the bending moment $(M_P + M_{0R})$ is

$$P_B = \tfrac{1}{2}\mathrm{Re}[-\,\mathrm{i}\omega(M_P + M_{0R})\theta_0^*]. \tag{23}$$

Flexural waves cause two internal forces (one associated with bending, the other with shear) to act in any beam element. For element CD, whose element coordinate system is xoy, the flexural displacement is

$$W(x) = \alpha_{x0'}M_L + \alpha_{xL'}M_L\mathrm{e}^{\mu}. \tag{24}$$

For harmonic motion, the flexural velocity and angular velocity are, respectively,

$$\dot{w}(x,t) = \mathrm{i}\omega W(x), \qquad \dot{\theta}(x,t) = \mathrm{i}\omega \partial W(x)/\partial x, \tag{25}$$

where a dot denotes $\partial/\partial t$.

The bending moment and shear force at any point x are, respectively,

$$M(x) = EI\partial^2 W(x)/\partial x^2, \qquad Q(x) = EI\partial^3 W(x)/\partial x^3. \tag{26}$$

The power flow $P_Q(x)$ and $P_M(x)$ transmitted, respectively, by shear force $Q(x)$ and bending moment $M(x)$ can be obtained in the form like equation (22), so the total power flow transmitted by internal forces is

$$P_T = P_Q(x) + P_M(x). \tag{27}$$

It can be proved that $P_T = P_C$ for undamped beam structures. For the same reason the total power flow transmitted by internal forces $Q(x)$ and $M(x)$ along the beam element AB in the negative directions equals P_B.

Based on the above analysis, the transmitted power flow is a function of position and the characteristic size of the damage. The power flow transmitted by internal forces $Q(x)$ and $M(x)$ can be easily measured with two-position linear accelerometers [13], so the damage can be diagnosed by comparing the vibrational power flow of beam structures with and without a damage.

6. CALCULATIONS AND DISCUSSION

In this paper, assume that the damaged element beam is known, which can simplify the analysis here and is a case in practical engineering. Further work will be done to solve the damage detection problem if the damaged element beam is not known previously.

For undamped periodic beam, it is found that there are alternate bands of propagation and stop of free waves. In stop bands, $\mu = \mu_r$ or $\mu = \mu_r - i\pi$ and then e^{μ} is real. So $P_B = P_C = 0$. In propagation bands, the propagation constant is in purely complex and frequency dependent. The transmitted power flow can be obtained by substitution of $\mu = i\mu_i$ into equations (22) and (23).

Because the damages are local phenomenon, the periodic condition has not changed. The propagation bands and the stop bands are the same of those of perfectly periodic beam. The lower and upper non-dimensional bounding frequencies of the first propagation band are 3·14 and 4·73, respectively, which are identical to the non-dimensional fundamental natural frequencies of a single beam when its ends are simply supported or fully clamped. The second non-dimensional bounding frequencies of the second propagation band are 6·28 and 7·85.

The total power P_P of periodically perfect beam without loss factor is independent of position, having the same values at the same λL and any station [14], and that is the case of $a = 0$ for damaged beam.

Tables 1 and 2 show the transmitted power flow ratio in the propagation bands when $a/2R = 0{\cdot}1$ and $a/2R = 0{\cdot}2$ respectively. Calculation results (above tables are just two samples) show that $P_C/P_B = 1{\cdot}0$ and the total power flow is less than that of perfect beam at any case. That is to say the total power P_T of periodically damaged beam has the same value at the same λL and any point too, though it is dependent of position and depth of the damage. If the ratio of depth of the damage to the diameter of the beam is small, the power flow ratio P_P/P_C is also small. It implies that the vibrational power flow of damaged beam is not sensitive to small damage. When the depth of the damage is fixed, the total power flow has mid-point symmetry to the position of the damage at the same dimensionless frequency λL. For example, when $\lambda L = 7{\cdot}0$ and $a/2R = 0{\cdot}2$, the $P_P/P_C = 1{\cdot}046$ for $x_0/L = 0{\cdot}3$ and $x_0/L = 0{\cdot}7$; when $\lambda L = 6{\cdot}5$ and $a/2R = 0{\cdot}1$, the $P_P/P_C = 1{\cdot}002$ for $x_0/L = 0{\cdot}2$ and $x_0/L = 0{\cdot}8$, etc.

Figure 3 shows how the power flow varies over the frequency range when $a/2R = 0{\cdot}4$, λL is the non-dimensional frequency, and $P_T L^2 \sqrt{S\rho}/(M_P\sqrt{EI}) \times 10^{-4}$ is the non-dimensional power flow. In even propagation band, every power flow curve decreases when λL increases. In odd propagation bands, the power flow is minimum when $x_0/L = 0{\cdot}5$ at the same λL, but it is maximum in even propagation bands.

Figure 4 shows how the power flow varies over the frequency range when $x_0/L = 0{\cdot}5$. In the first propagation band, the power flow decreases when $a/2R$ increases, because the increasing rotation stiffness caused by the damage makes the mobility at both sides of the damage point much more unmatchable. So more of incidence wave is reflected and less is

TABLE 1

The transmitted power flow ratio ($a/2R = 0{\cdot}1$)

	$x_0/L = 0{\cdot}2$		$x_0/L = 0{\cdot}5$		$x_0/L = 0{\cdot}8$	
λL	P_P/P_C	P_C/P_B	P_P/P_C	P_C/P_B	P_P/P_C	P_C/P_B
3·25	1·000	1·000	1·001	1·000	1·000	1·000
3·50	1·000	1·000	1·000	1·000	1·000	1·000
3·75	1·000	1·000	1·000	1·000	1·000	1·000
4·00	1·000	1·000	1·000	1·000	1·000	1·000
4·25	1·000	1·000	1·000	1·000	1·000	1·000
4·50	1·000	1·000	1·001	1·000	1·000	1·000
6·50	1·002	1·000	1·000	1·000	1·002	1·000
6·75	1·001	1·000	1·000	1·000	1·001	1·000
7·00	1·001	1·000	1·000	1·000	1·001	1·000
7·25	1·001	1·000	1·000	1·000	1·001	1·000
7·50	1·001	1·000	1·000	1·000	1·001	1·000
7·75	1·001	1·000	1·000	1·000	1·001	1·000

TABLE 2

As Table 1, *but* ($a/2R = 0{\cdot}2$)

	$x_0/L = 0{\cdot}3$		$x_0/L = 0{\cdot}5$		$x_0/L = 0{\cdot}7$	
λL	P_P/P_C	P_C/P_B	P_P/P_C	P_C/P_B	P_P/P_C	P_C/P_B
3·25	1·020	1·000	1·045	1·000	1·020	1·000
3·50	1·007	1·000	1·016	1·000	1·007	1·000
3·75	1·005	1·000	1·012	1·000	1·005	1·000
4·00	1·004	1·000	1·012	1·000	1·004	1·000
4·25	1·003	1·000	1·015	1·000	1·003	1·000
4·50	1·003	1·000	1·026	1·000	1·003	1·000
6·50	1·080	1·000	1·0006	1·000	1·080	1·000
6·75	1·051	1·000	1·0011	1·000	1·051	1·000
7·00	1·046	1·000	1·0014	1·000	1·046	1·000
7·25	1·051	1·000	1·0015	1·000	1·051	1·000
7·50	1·071	1·000	1·0013	1·000	1·070	1·000
7·75	1·198	1·000	1·0006	1·000	1·198	1·000

transmitted. In the second propagation band, though the power flow decreases when $a/2R$ increases too, there are small differences between curves. This is an important property of damaged beam.

7. CONCLUSIONS

In this report a theoretically sound computational method that can be a basis of damage diagnosis in periodic beam has been successfully presented based on a non-modal framework. The damage is modelled as a joint of a local spring. The damage point transfer matrix and the element transfer matrix are deduced. In the view of structure-borne sound, vibrational power flow analysis is investigated for damaged beam structures. Then the

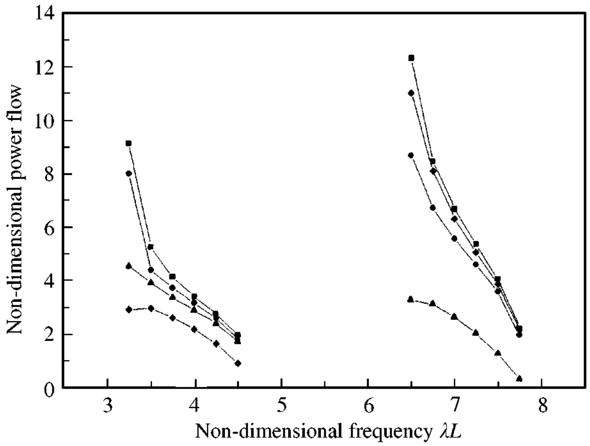

Figure 3. Transmitted power flow when $a/2R = 0{\cdot}4$ for damaged beam: —●— $x_0/L = 0{\cdot}1$; —▲— 0·3; —◆— 0·5; —■— perfect beam.

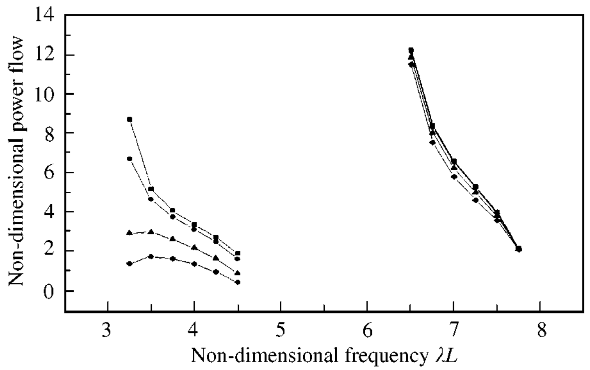

Figure 4. Transmitted power flow when $x_0/L = 0{\cdot}5$ for damaged beam: —■— $a/2R = 0{\cdot}2$; —●— 0·3; —▲— 0·4; —◆— 0·6.

relations of the vibrational power flow, the position and the characteristic size of the damage are obtained combined with periodic structure theory, and some important remarks are obtained. Based on the above analysis and measurement results, the damage can be diagnosed by comparing the vibrational power flow of beam structures with and without a damage in the next work.

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